On the Representation of Functions in Series of the Form $\Sigma c_n g(x+n)$.*

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Introduction.

The most important functions defined by the Ω- and $\bar{\Omega}$-series, whose properties I have investigated in previous memoirs,† are doubtless those which have in a half-plane a Poincaré asymptotic representation ‡ in descending power series; and, in particular, those in which the series depend on a defining function $g(x)$ which has the asymptotic form §

$$g(x) \sim x^{\mu-x} e^{a+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right), \tag{1}$$

valid in a sector V including the positive axis of reals in its interior, and which is analytic in V for sufficiently large values of $|x|$. We shall now use the symbol $S(x)$ to denote the series

$$S(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \tag{2}$$

It will be observed that $S(x)$ belongs to the class of series denoted by $\bar{\Omega}(x)$ in the preceding papers, and also to the more special class denoted by $\bar{\omega}(x)$.

In the present paper I make a contribution towards solving the problem || of representing given functions in the form of series $S(x)$ depending on a

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† *Transactions American Mathematical Society*, Vol. XVII (1916), pp. 207–232; American Journal of Mathematics, Vol. XXXIX (1917), pp. 385–403. These papers will be referred to by the numbers I and II, respectively.

‡ Compare II, especially §§ 4 and 5.

§ It is convenient to introduce here a slight change in notation; it is one, however, which can cause no confusion.

|| That this includes the fundamental problem of representing given functions in the form of series dependent on functions defined by linear homogeneous difference equations may be seen from the following considerations. A fundamental property of the leading functions $G(x)$ defined by such equations is that expressed in the asymptotic relation

$$G(x) \sim x^{ix} e^{\beta x}\left(1+\frac{b_1}{x}+\frac{b_2}{x^2}+\ldots\right)$$

valid in a right half-plane, i being an integer and $\beta, b_1, b_2, \ldots$ being constants. (Compare Birkhoff, *Transactions American Mathematical Society*, Vol. XII (1911), pp. 243–284; and Carmichael, *ibid.*, pp.

given defining function $g(x)$. In order to render this problem amenable to simple methods it is necessary to place further restrictions on $g(x)$. We have seen (II, § 5) that

$$\frac{g(x+n)}{g(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots\right), \quad \beta_{0n}=e^{(\beta-1)n}, \tag{3}$$

where $\beta_{0n}, \beta_{1n}, \beta_{2n}, \ldots$ are a set of numbers independent of x. From this it follows that we have an asymptotic relation of the form

$$\frac{g(x)}{g(x+n)} \sim x^{n}\left(\gamma_{0n}+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots\right). \tag{4}$$

Some of the restrictions mentioned are stated most conveniently in terms of the coefficients γ. They appear in Section 4. Others appear in Section 5.

In § 1 I derive, in simple form, a necessary and sufficient condition on the coefficients c_n implying the convergence of $S(x)$ in an appropriately determined right half-plane. In § 2 some remarks are made on the order of increase of the coefficients in certain Poincaré asymptotic series. In § 3 the fundamental relations between these coefficients and the coefficients c_n of the associated series $S(x)$ are determined. In § 4 I construct, in a particular manner, functions having given Poincaré asymptotic representations of a certain type, and incidentally point out some fundamentally important instances of the series $S(x)$ which have occurred in the recent literature. Finally, in § 5, I show how to transform the Borel integral sum of a divergent series into a series $S(x)$ and also into a certain natural generalization of such series, and indicate some wide ranges of applicability of these results.

§ 1. *Order of Increase of Coefficients c_n in a Converging Series $S(x)$.*

Let $S(x)$ be a series of the form (2) which converges at every non-exceptional point in some half-plane; that is, one whose convergence number is not $-\infty$. Then, from the corollary to Theorem XII, in Memoir I, it follows that a real constant r_1 (finite or equal to $-\infty$) exists such that

$$\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi}=r_1, \tag{5}$$

99–134, and AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1916), pp. 163–182). In case $i=-1$ the function $G(x)$ itself is of the same type as the function $g(x)$ in the text. In general a suitable value of $g(x)$ may be defined in terms of $G(x)$ by the relation

$$g(x)=\frac{G(x)}{[\Gamma(x)]^{i+1}},$$

or by any one of several similar expressions which readily come to mind. Thus, for every function of the type $G(x)$ there exist corresponding functions of the type $g(x)$ suitable for use in the formation of series $S(x)$.

where $\Sigma_\xi \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^\xi, \tag{6}$$

$E(\xi)$ denoting the greatest integer not greater than ξ. From the last inequality we see that

$$\xi < E(\xi) + 1 \leqq 1 + \log \nu$$

for every positive value of ξ and corresponding value of ν. Hence, from (5), it follows that a constant r_2 exists such that

$$\log |c_\nu g(\nu)| < r_2(1+\log \nu), \qquad \nu > 0.$$

Thence it follows readily that a constant r exists such that

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2. \tag{7}$$

Relation (7) states a condition which is necessary if $S(x)$ is to converge at every non-exceptional point in some half-plane.

It may also be shown that (7) states a condition sufficient to ensure that $S(x)$ thus converges in some half-plane. For from (7) it is clear that

$$\Sigma_\xi |c_\nu g(\nu)| < \nu^\rho e^\xi,$$

where ν and ξ are related as in (6), and ρ is a positive number not less than r. From this one sees readily that the superior limit in the first member of (5) has a value different from $+\infty$. This, in connection with the corollary to Theorem XII, in Memoir I, at once yields the conclusion that $S(x)$ converges at every non-exceptional point in some half-plane.

Thus we are led to the following theorem:*

THEOREM I. *A necessary and sufficient condition that the series $S(x)$ in (2) shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2.$$

In view of the asymptotic form of $g(\nu)$ this theorem is seen to be equivalent to the following corollary:

COROLLARY: *A necessary and sufficient condition that the series $S(x)$ in (2) shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu| < \nu^\nu e^{-\nu R(\beta)} \nu^r, \qquad \nu \geqq 2.$$

* By exactly the same argumentation one is led to precisely the same necessary and sufficient condition for the similar convergence of the series $\Omega(x)$ and $\bar{\Omega}(x)$ in the case when $k=1$ and $m=0$ or 1. The corollary to Theorem XII, in Memoir I, yields immediately the corresponding result for the case when k and m do not satisfy the conditions just stated.

It is easy to see that an equivalent statement is obtained if one replaces the inequality in this corollary by the following:

$$|c_\nu| < \nu!\, e^{\nu[1-R(\beta)]}\nu^r, \qquad \nu \geqq 2.$$

To prove this, one has only to employ the well-known fact that

$$(\nu+1)!\, \nu^{-\nu} e^{\nu} \nu^{\frac{1}{2}}$$

approaches a finite limit different from zero as ν approaches infinity.

§ 2. *Order of Increase of the Coefficients in the Poincaré Asymptotic Representation of $S(x)$.*

We have seen (II, Theorem II and § 5) that a function $S(x)$, defined by the series in (2), has a Poincaré asymptotic representation of the form

$$S(x) \sim \beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \dots . \tag{8}$$

valid in that part of V which lies in the half-plane $R(x) \geqq -\lambda_1$, where λ_1 is less than the convergence number λ of $S(x)$, and that the coefficients β_ν are given by the formulae

$$\beta_\nu = c_0\beta_{\nu 0} + c_1\beta_{\nu-1,1} + c_2\beta_{\nu-2,2} + \dots + c_\nu\beta_{0\nu}, \qquad \nu = 0, 1, 2, \dots . \tag{9}$$

The order of increase of $|\beta_\nu|$ with respect to ν evidently depends upon the β_{ij} quite as much as upon the c_i. In case the β_{ij} satisfy the restrictive condition

$$|\beta_{\nu-k,k}| < M e^{kR(\beta)} \nu^{\nu+\sigma} k^{-k}, \tag{10}$$

it is easy to see (by aid of the corollary to Theorem I) that we have

$$|\beta_\nu| < \nu^\nu \nu^s, \qquad \nu \geqq 2, \tag{11}$$

s being an appropriately chosen quantity independent of ν.

For the special case in which $g(x+1)/g(x)$ is analytic at $x=\infty$, other inequalities similar to (10) and (11) may be obtained. We write

$$\frac{g(x+1)}{g(x)} \equiv \rho(x) = \frac{\rho_1}{x} + \frac{\rho_2}{x^2} + \frac{\rho_3}{x^3} + \dots .$$

Since this series converges for some x, a positive quantity ρ exists such that

$$|\rho_r| < \rho^r.$$

We take ρ to be greater than unity. We may write,

$$\frac{g(x+n)}{g(x)} \equiv \rho(x)\rho(x+1)\dots\rho(x+n-1) = \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x+i} + \frac{\rho_2}{(x+i)^2} + \dots\right]$$

$$= \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x} + \frac{\rho_2 - \rho_1 i}{x^2} + \frac{\rho_3 - 2\rho_2 i + \rho_1 i^2}{x^3} + \dots\right].$$

If this last product is expanded in powers of $1/x$, it is easy to see that the coefficients in the resulting series are less in absolute value than the coefficients in the similar expansion of the product

$$\prod_{i=0}^{n-1}\left[\frac{\rho}{x}+\frac{\rho(\rho+i)}{x^2}+\frac{\rho(\rho+i)^2}{x^3}+\frac{\rho(\rho+i)^3}{x^4}+\ldots\right].$$

These coefficients are not decreased if in this product $\rho(\rho+i)^{m-1}$ is replaced by $(\rho+n)^m$. Thence, by the multinomial theorem, it follows readily that

$$|\beta_{\nu-n,n}|<(n+\rho)^\nu\Sigma\frac{(i_1+i_2+\ldots+i_n)!}{i_1!\,i_2!\ldots i_n!},$$

where the summation is taken subject to the condition

$$i_1+2i_2+3i_3+\ldots+ni_n=\nu;$$

whence it follows that

$$|\beta_{\nu-n,n}|<\nu^n(n+\rho)^\nu. \tag{12}$$

Thence, through (9), and the corollary to Theorem I, we see that

$$|\beta_\nu|<\nu^{3\nu}e^{-\nu R(\beta)}\nu^t, \qquad \nu\geqq 2, \tag{13}$$

t being an appropriately chosen real quantity.

§ 3. *Evaluation of the Constants c_ν in Terms of the Constants β_ν.*

From (9) it follows that the constants c_ν may be expressed directly in terms of the constants β_ν. It is more convenient for our purposes, however, to proceed as follows: In equation (4) replace n by $\nu-n$, where $n<\nu$, and then replace x by $x+n$. Thus we have

$$\frac{g(x+n)}{g(x+\nu)}\sim(x+n)^{\nu-n}\left(\gamma_{0,\nu-n}+\frac{\gamma_{1,\nu-n}}{x+n}+\frac{\gamma_{2,\nu-n}}{(x+n)^2}+\ldots\right). \tag{14}$$

Then from

$$\frac{g(x)}{g(x+\nu)}\left(\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots\right)\sim\sum_{n=0}^{\infty}c_n\frac{g(x+n)}{g(x+\nu)},$$

by means of (4) and (14), we have

$$\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\ldots+\gamma_{\nu+1,\nu}\beta_0=\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}$$
$$+\gamma_{32}c_{\nu-2}+\ldots+\gamma_{\nu+1,\nu}c_0,$$

on equating the coefficients of $1/x$ in the expanded asymptotic forms of the two members. Now it is easy to see that

$$c_0=\beta_0,\quad c_1=\beta_1. \tag{15}$$

Hence, if we write

$$\eta_\nu=\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\ldots\ldots+\gamma_{\nu\nu}\beta_1,\ \nu>0;\qquad \eta_0=\beta_1, \tag{16}$$

we have

$$\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}+\gamma_{32}c_{\nu-2}+\ldots\ldots+\gamma_{\nu,\,\nu-1}c_1=\eta_\nu,\qquad \nu=1,\,2,\,\ldots\ldots.$$

Combining this system with the second equation in (15), and solving for $c_{\nu+1}$ we have

$$c_{\nu+1}=\gamma_{01}\eta_\nu+\gamma_{01}^3\eta_{\nu-2}\begin{vmatrix}0 & \gamma_{21}\\ \frac{1}{\gamma_{01}} & 0\end{vmatrix}-\gamma_{01}^4\eta_{\nu-3}\begin{vmatrix}0 & \gamma_{21} & \gamma_{32}\\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21}\\ 0 & \frac{1}{\gamma_{01}} & 0\end{vmatrix}$$

$$+\gamma_{01}^5\eta_{\nu-4}\begin{vmatrix}0 & \gamma_{21} & \gamma_{32} & \gamma_{43}\\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21} & \gamma_{32}\\ 0 & \frac{1}{\gamma_{01}} & 0 & \gamma_{21}\\ 0 & 0 & \frac{1}{\gamma_{01}} & 0\end{vmatrix}-\ldots\ldots,\qquad \nu=1,\,2,\,3,\,\ldots\ldots, \tag{17}$$

the expansion ending with the term containing η_0. Here γ_{01} has the value

$$\gamma_{01}=e^{1-\beta}.$$

We shall use the symbol Δ_k for the determinant of order k in the second member of equation (17).

§ 4. *Construction of Functions Having Given Poincaré Asymptotic Representations of a Certain Type.*

In this section we show how to construct a function $f(x)$ having a given Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots\ldots, \tag{18}$$

of a certain type. We confine our attention to the case in which the function $g(x)$ and the constants $\beta_0,\,\beta_1,\,\beta_2,\,\ldots\ldots$ are jointly subject to the condition that positive quantities M and σ, both independent of ν, exist such that

$$|\eta_\nu|<M\nu^\nu e^{-\nu R(\beta)}\nu^\sigma,\qquad \nu=1,\,2,\,3,\,\ldots\ldots, \tag{19}$$

η_ν having the definition given in (16). Moreover, we suppose that $g(x)$ is such that a positive quantity M_1, and a real non-negative quantity σ_1, both independent of k, exist such that

$$|\Delta_k|<M_1k^ke^{-k}k^{\sigma_1}, \tag{20}$$

where Δ_k denotes the determinant defined at the end of Section 3. Our central theorem here is the following:

THEOREM II. *Let* $\beta_0, \beta_1, \beta_2, \ldots,$ *be a given set of constants, and let* $g(x)$ *be an associated function such that relations* (19) *and* (20) *are satisfied. Then, if constants* $c_0, c_1, c_2, \ldots$ *are determined in terms of* $\beta_0, \beta_1, \beta_2, \ldots,$ *by means of equations* (15) *and* (17), *the series*

$$\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)} \tag{21}$$

converges absolutely at all non-exceptional points in the half-plane $R(x) > \tau+1$, *where* τ *is the greater of the two quantities* 0 *and* $\sigma+\sigma_1+R(\mu)$; *and the function* $f(x)$ *represented by this series satisfies the Poincaré asymptotic relation* (18), *the latter being valid in the greatest region in the x-plane which is common to V and the half-plane* $R(x) \geqq \tau+1+\varepsilon$, ε *being any positive constant.*

We prove first that part of the conclusion which relates to the convergence of the series in (21). From relations (17), (19), (20) we see readily that

$$|c_{\nu+1}| < M_2 \nu^{\nu} e^{-\nu R(\beta)} \nu^{\sigma+\sigma_1+1}, \tag{22}$$

where M_2 is a quantity independent of ν. From the corollary to Theorem I it follows now that the series in (21) is convergent in an appropriately chosen right half-plane. Moreover, the last inequality affords partial information as to the position of the line of absolute convergence. This may be shown as follows: From the corollary to Theorem XII, in Memoir I, we see that the abscissa μ of absolute convergence of the series in (21) is given by the relation

$$\mu = -\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi},$$

where $\Sigma_\xi \zeta_\nu$ stands for the sum of all ζ_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^{\xi}.$$

From (22) and the asymptotic form of $g(\nu)$ it is easy to see that

$$|c_\nu g(\nu)| < M_3 \nu^{\sigma+\sigma_1+R(\mu)}, \qquad \nu = 2, 3, \ldots,$$

M_3 being an appropriately chosen constant. Hence we have

$$\Sigma_\xi |c_\nu g(\nu)| < M_3 \sum_{\nu=1}^{\eta_\xi} \nu^{\sigma+\sigma_1+R(\mu)} \leqq M_3 \eta_\xi^{\tau+1},$$

where η_ξ is the greatest value of ν, such that $\log \nu < \xi$ and τ is the greater of the quantities 0 and $\sigma+\sigma_1+R(\mu)$. Hence,

$$\mu \geqq -\limsup_{\xi=\infty} \frac{(\tau+1)\log \eta_\xi + \log M_3}{\xi} = -(\tau+1).$$

This completes the proof of the part of the theorem which refers to the convergence of the series in (21).

The portion of the conclusion of the foregoing theorem which relates to the asymptotic character of $f(x)$ is an immediate consequence of Theorem II, of Memoir II, and the convergence properties just established.

The foregoing theorem states conditions to ensure the convergence of the series in (21). A central part of these conditions is contained in relations (19). If $g(x)$ satisfies conditions associated with (1), and if, moreover, $g(x)/g(x+\nu)$ is a polynomial when ν is sufficiently large, then from (17), and the corollary to Theorem I, it is easy to see that *relation* (19) *affords a necessary and sufficient condition for the convergence of* (21); and hence for the construction, by means of a series (21), of a function $f(x)$ having the given Poincaré asymptotic representation (18). It is not difficult to determine other classes of functions $g(x)$ for which (19) plays a like rôle in respect to necessary and sufficient conditions.

An important special use of the result contained in Theorem II should be pointed out. In many investigations, particularly in the theory of differential equations and of difference equations, one is led to divergent power series when one seeks to obtain a suitable representation of a function which is to be determined. If for an appropriately chosen function $g(x)$ the coefficients $\beta_0, \beta_2, \beta_3, \ldots,$ in the diverging power series thus arising satisfy the conditions imposed by relations (19), then it is clear that a suitable modification of the computations will enable one to obtain a convergent expansion (21) in place of the diverging power series. It may be anticipated with considerable confidence that this converging expansion will serve to define such a function as one is seeking. It is my intention later to present such applications of Theorem II as are here indicated. For the special class of factorial series, important applications of this character have already been made in the theory both of differential equations * and of difference equations,† and in more general aspects of the theory of functions. ‡

We shall next point out some simple conditions which are sufficient to ensure that relations (19) and (20) shall be satisfied. In the treatment of these conditions we shall have need of Hadamard's fundamental theorem § concerning an upper bound to the absolute value of a determinant. This theorem may be stated as follows:

* Horn, *Mathematische Annalen*, Vol. LXXI (1912), pp. 510–532.

† Nörlund, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXV (1913), pp. 177–216.

‡ Watson, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXIV (1912), pp. 41–48.

§ Hadamard, *Bulletin des Sciences Mathématiques* (Darboux), Vol. XVII (1893), pp. 240–246.

If Δ is a determinant of order n in which a_{ij} is the element in the *i-th* row and the *j-th* column, then

$$|\Delta| \leqq \sqrt{r_1 r_2 \ldots . r_n}, \quad |\Delta| \leqq \sqrt{\rho_1 \rho_2 \ldots . \rho_n},$$

where

$$r_i = \sum_{j=1}^{n} |a_{ij}|^2, \quad \rho_i = \sum_{j=1}^{n} |a_{ji}|^2.$$

With this theorem in hand we may readily determine conditions implying relation (20). Thus, if $\gamma_{k+1,k}$ satisfies the condition

$$|\gamma_{k+1,k}| \leqq k^{\frac{1}{2}}, \qquad k = K, K+1, K+2, \ldots ., \tag{23}$$

where K is a fixed integer, it may easily be shown that (20) is satisfied. For, from the first inequality in Hadamard's theorem above, we see that

$$|\Delta_k| < \bar{M} \cdot k! = \bar{M} \cdot \Gamma(k+1),$$

$\bar{M}$ being an appropriately chosen quantity independent of k. By means of the well-known asymptotic formula for $\Gamma(x)$, namely,

$$\Gamma(x) \sim x^x e^{-x} x^{-\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \ldots .\right),$$

it is now easy to see that (20) is satisfied provided that M_1 and σ_1 are given appropriate values. Hence, relation (23) expresses a condition sufficient *to ensure that a relation of the form* (20) *is satisfied.*

In order to determine simple conditions ensuring that (19) is satisfied we observe that the asymptotic relation (1) implies that the quantity

$$\frac{g(\rho+\nu)}{g(\rho)} \nu^{\nu} e^{-\beta\nu} \nu^{\rho-}$$

approaches a finite value different from zero as n approaches infinity, provided that $x = \rho$ is a point at which $g(x)$ is analytic and different from zero. From this it follows at once that a relation of the form (19) is satisfied, provided that a positive quantity M' and a non-negative real quantity σ', both independent of ν, exist such that

$$|\eta_\nu| < M' \left| \frac{g(\rho)}{g(\rho+\nu)} \right| \nu^{\sigma'}. \tag{24}$$

Let us consider now the special case in which the series in (18) converges when $|x|$ is sufficiently large. Then through (16) we see that a positive number ρ exists such that

$$\frac{1}{\rho} |\eta_\nu| < |\gamma_{0\nu}| \rho^{\nu} + |\gamma_{1\nu}| \rho^{\nu-1} + |\gamma_{2\nu}| \rho^{\nu-2} + \ldots . + |\gamma_{\nu-1,\nu}| \rho + |\gamma_{\nu\nu}|,$$

$$\nu = 1, 2, 3, \ldots ..$$

Comparing this with (4) we see that condition (24) is obviously satisfied in the special case when the quantities $\gamma_{0n}, \gamma_{1n}, \gamma_{2n}, \ldots$ are all positive or zero for every positive integral value of n greater than some given value N. Other simple conditions under which (24) is satisfied will readily occur to the reader.

Again, it may be seen that a relation of the form (19) is satisfied in case a positive quantity M and a non-negative real quantity σ exist such that

$$|\gamma_{k\nu}\beta_{\nu-k+1}| < M\nu^{\nu}e^{-\nu R(\beta)}\nu^{\sigma-1} \tag{25}$$

for every ν and corresponding k not greater than ν. For determining whether (25) is satisfied one may make use of the relation

$$\gamma_{k\nu} = \frac{1}{2\pi i}\int_C \frac{f_\nu(x)\,dx}{x^{\nu-k+1}}, \tag{26}$$

where

$$f_\nu(x) = \gamma_{0\nu}x^{\nu} + \gamma_{1\nu}x^{\nu-1} + \ldots + \gamma_{\nu\nu}$$

and C is a circle about the point $x=0$ as a center. From this it follows at once that

$$|\gamma_{k\nu}|\rho^{\nu-k} \leq \frac{1}{2\pi\rho} M_\rho[f_\nu(x)], \tag{27}$$

where $M_\rho[f_\nu(x)]$ denotes the maximum value of $|f_\nu(x)|$ on the circle of radius ρ about the point $x=0$ as a center. There are large classes of cases in which it may readily be shown that relation (25) is implied by relation (27).

Let us consider the application of these results in the case of factorial series. Here $g(x) = 1/\Gamma(x)$; and we have

$$\frac{g(x)}{g(x+n)} = x(x+1)(x+2)\ldots(x+n).$$

It is obvious that the quantities γ_{sn} are all positive or zero, and that they are zero in case $s \geq n$. Moreover,

$$\gamma_{0\nu}\rho^{\nu} + \gamma_{1\nu}\rho^{\nu-1} + \ldots + \gamma_{\nu\nu} = \frac{g(\rho)}{g(\rho+\nu)}.$$

Hence, conditions (23) and (24) are satisfied in case (18) converges for sufficiently large values of $|x|$, say for $|x| \geq R$. Moreover, for this special case it is clear that we may take $\sigma_1 = 0$ and $\sigma = R - \mu$ and that μ now has the value $\frac{1}{2}$. Then it is easy to see that the line of absolute convergence of the series in (21), in this case cuts the axis of reals at a point not further to the right than a unit's distance to the right of the rightmost point of the circle of convergence of the power series in $1/x$ by which $f(x)$ may be represented.

Again, if we take for $g(x)$ the value

$$g(x)=\frac{1}{a^x\Gamma\left(x+\dfrac{b}{a}\right)},\tag{28}$$

we have

$$\frac{g(x)}{g(x+1)}=ax+b.$$

Then, as in the preceding case, it is easy to see that the conditions of Theorem II are satisfied provided that (18) converges for sufficiently large $|x|$ and a is positive while b is positive or zero. If we put $b=0$, series (1) obviously takes the special form

$$S_1(x)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{ax(ax+a)(ax+2a)\dots(ax+\overline{n-1}a)}.$$

Replacing ax by z, we may write this in the form

$$\overline{S}_a(z)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{z(z+a)(z+2a)\dots(z+[n-1]a)}.\tag{29}$$

Series of this class play a leading rôle in Nörlund's fundamental paper on factorial series to which reference has already been made.

Again, if we put Mz for x and $\alpha+1$ for b/a in the function $g(x)$ of equation (28) the corresponding series $S(x)$ is transformed into the form

$$b_0+\frac{b_1}{Mz+\alpha+1}+\frac{b_2}{(Mz+\alpha+1)(Mz+\alpha+2)}$$
$$+\frac{b_3}{(Mz+\alpha+1)(Mz+\alpha+2)(Mz+\alpha+3)}+\dots,\tag{30}$$

a series which plays the leading rôle in the important paper of Watson's referred to above. Watson exhibits a large class of functions expansible in converging series (30).

One may easily construct many other particular functions $g(x)$ satisfying those hypotheses of Theorem II which relate to $g(x)$ provided that the associated series (18) converges for sufficiently large $|x|$. Some of the most interesting of such functions $g(x)$ are readily expressible in terms of the gamma function. Such a one, for instance, is the function $g(x)$,

$$g(x)=\frac{\Gamma(x+1)}{\Gamma(x)\Gamma(x+3)}.\tag{31}$$

Here we have

$$\frac{g(x)}{g(x+1)}=\frac{x(x+3)}{x+1}.$$

Employing the relation

$$\frac{g(x)}{g(x+n)}=\frac{g(x)}{g(x+1)}\cdot\frac{g(x+1)}{g(x+2)}\dots\frac{g(x+n-1)}{g(x+n)},$$

it is thence easy to see that $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large. Hence, relations (23) and (24) are satisfied.

In a similar way one may treat the function

$$g(x)=\frac{\Gamma(x+1)\Gamma(x+3)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)}. \tag{32}$$

It is clear that one may thus form an indefinitely great number of similar functions $g(x)$ such that in each case $g(x)/g(x+1)$ is a rational function of x, and $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large.

§ 5. *Representation of Given Functions in the Form of Convergent Series $S(x)$.*

By means of Borel's method of summation of series (in general divergent) we shall now show how to represent functions of a certain important class in the form of convergent series $S(x)$. The functions treated are those obtained by taking the sum of a series of the form

$$\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\frac{\beta_3}{x^2}+\ldots. \tag{33}$$

by means of Borel's integral method of summation. Denoting the Borel integral sum of the series in (33) by $f(x)$, we have by definition:*

$$f(x)=\int_0^\infty xe^{-tx}\phi(t)\,dt, \tag{34}$$

where

$$\phi(t)=\beta_0+\beta_1 t+\frac{\beta_2 t^2}{2!}+\frac{\beta_3 t^3}{3!}+\ldots+\frac{\beta_n t^n}{n!}+\ldots. \tag{35}$$

We assume † that the series in (35) is convergent for all values of t, and that the function $\phi(t)$ defined by it is such that the integral in (34) exists. We say then that the series (34) is summable to the sum $f(x)$.

In this section we shall suppose † also that the asymptotic series in (3) is summable in the sense of Borel, so that we have

$$\frac{g(x+n)}{g(x)}=\int_0^\infty xe^{-tx}\phi_n(t)\,dt, \qquad n=1,\,2,\,3,\,\ldots, \tag{36}$$

where

$$\phi_n(t)=\frac{\beta_{0n}t^n}{n!}+\frac{\beta_{1n}t^{n+1}}{(n+1)!}+\frac{\beta_{2n}t^{n+2}}{(n+2)!}+\ldots. \tag{37}$$

The series in (37) we shall take to be convergent for all values of t.

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* Borel, "Leçons sur les séries divergentes," p. 108 ff. We have made certain obvious reductions so as to obtain the form convenient to use with descending series (33) rather than with ascending series.

† The restrictions made here are stronger than are essential to the argument. They may be weakened in accordance with certain general considerations mentioned by Borel (*loc. cit.*, p. 99).

Of the function $f(x)$ defined in (34) it is easy to obtain a *formal* expansion in series $S(x)$. For this purpose we note that by means of equation (9) it is easy to establish the formal relation

$$\beta_0+\beta_1 t+\ldots.+\frac{\beta_n t^n}{n!}+\ldots.=\sum_{n=0}^{\infty} c_n \sum_{\nu=n}^{\infty} \beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty} c_n \phi_n(t). \tag{38}$$

Replacing $\phi(t)$ in (34) by the second member of the last equation we have the formal relation

$$f(x)=\int_0^\infty\left(\sum_{n=0}^{\infty} c_n x e^{-tx}\phi_n(t)\right)dt. \tag{39}$$

If, still proceeding formally, we integrate term by term the series denoted by the summation in (39) and make use of (36), we have

$$f(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \tag{40}$$

Hence,

THEOREM III. *In all cases in which* (36) *and* (38) *are valid relations, and the series denoted by the outer summation in* (39) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (40) *in the form of a converging series* $S(x)$.

This result is capable of a ready generalization as follows: From (34) we have

$$f(ax)=\int_0^\infty axe^{-atx}\phi(t)\,dt=\int_0^\infty xe^{-tx}\psi(t)\,dt, \tag{41}$$

where

$$\psi(t)=\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\frac{\beta_3}{a^3}t^3+\ldots..$$

Analogous to (9) we now form the equations

$$\frac{\beta_\nu}{a^\nu}=\bar{c}_0\beta_{\nu 0}+\bar{c}_1\beta_{\nu-1,1}+\bar{c}_2\beta_{\nu-2,2}+\ldots.+\bar{c}_\nu\beta_{0\nu}, \qquad \nu=0,1,2,\ldots.,$$

thus introducing the quantities $\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots..$ Thence proceeding formally, we have

$$\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\ldots.=\sum_{n=0}^{\infty}\bar{c}_n\sum_{\nu=n}^{\infty}\beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty}\bar{c}_n\phi_n(t); \tag{42}$$

whence, as before, we obtain

$$f(ax)=\int_0^\infty\left(\sum_{n=0}^{\infty}\bar{c}_n x e^{-tx}\phi_n(t)\right)dt. \tag{43}$$

Integrating term by term, we have

$$f(ax) = \sum_{n=0}^{\infty} \bar{c}_n \frac{g(x+n)}{g(x)};$$

or

$$f(x) = \sum_{n=0}^{\infty} \bar{c}_n \frac{g(x/a+n)}{g(x/a)}. \tag{44}$$

We are thus lead to the following result:

THEOREM IV. *In all cases in which* (36) *and* (42) *are valid relations and the series denoted by the summation in* (43) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (44) *in the form of a converging series into which a series* $S(x)$ *is readily transformed.*

If we take for $g(x)$ the particular value

$$g(x) = \frac{1}{a^x \Gamma(x)},$$

then relation (44) takes the special form

$$f(x) = \bar{c}_0 + \frac{\bar{c}_1}{x} + \frac{\bar{c}_2}{x(x+a)} + \frac{\bar{c}_3}{x(x+a)(x+2a)} + \frac{\bar{c}_4}{x(x+a)(x+2a)(x+3a)} + \cdots\cdots \tag{45}$$

That is to say, the Borel sum $f(x)$ of (33) is always represented formally by the series in (45). In case (33) is absolutely and uniformly summable by the integral method of Borel to the sum $f(x)$, then Nörlund (*loc. cit.*, p. 379) has shown that $f(x)$ has an actual convergent expansion (45) in case a is a sufficiently large positive number dependent upon $f(x)$. Hence the formal result in (45), and therefore the more general one in (44), has a wide range of useful applicability.

For special functions $g(x)$ (of which that in the preceding paragraph is an example) and corresponding classes of functions $f(x)$ it is possible, as we have just seen, to state more explicit and precise results than those obtained in Theorems III and IV; but it seems to be difficult to render these theorems more precise without further restrictions on $g(x)$. I propose as an important problem the determination of classes of functions $g(x)$, and corresponding classes of functions $f(x)$, such that the representations (40) and (44) are valid, either separately or simultaneously.

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